Unobserved Heterogeneous Spillover Effects in Instrumental Variable Models

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Outline

Motivation: Spillover Effects

- **Spillovers**: Each unit's outcome depends on others' treatment ($D_{-i} \longrightarrow Y_i$)

- Violation of SUTVA
  - Y_i depends only on D_i, not on D_{-i}

- Further complication: Treatment D_i may be **endogenous**

- Goal: Study causal effects when there are spillovers and treatment is endogenous

Motivation: An Example

Spillovers within best-friend groups; no spillovers across groups

- Researchers study how college completion (D_i) affects later earnings (Y_i)
- **Spillovers**: Best friend's college completion (D_{-i}) $\rightarrow$ individual's earnings (Y_i)

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- **Spillovers**: Best friend's college completion (D_{-i}) $\longrightarrow$ individual's earnings (Y_i)
- **Endogeneity**: College decisions are not random
  - Individual's choice depends on unobserved traits (V_i) that also affect earnings

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- **Spillovers**: Best friend's college completion (D_{-i}) $\longrightarrow$ individual's earnings (Y_i)
- **Endogeneity**: College decisions are not random
  - Individual's choice depends on unobserved traits (V_i) that also affect earnings
- **Heterogeneity**: The effects may differ across individuals
  - Spillover effect ($D_{-i} \to Y_i$) and direct effect ($D_i \to Y_i$) may vary with unobserved V_i

Treatment Effects With Heterogeneity: SUTVA case

Under SUTVA: $Y_i = Y_i(D_i)$

$$\begin{tikzcd} & V_i \arrow[dl, dashed] \arrow[dr, dashed] & \\ D_i \arrow[rr] & & Y_i \end{tikzcd}$$

Treatment effect **varies with unobserved trait** V_i

$$\mathbb{E}[Y_i(1) - Y_i(0) \mid V_i]$$

- V_i in a region: Local average treatment effect (LATE, Imbens & Angrist, 1994)
    - The return to education for individuals induced to complete college by the instrument
- V_i at a given value: Marginal treatment effect (MTE, Heckman & Vytlacil, 1999, 2001, 2005)
    - The return to education for individuals with a given level of unobserved ability

Treatment Effects With Heterogeneity: Spillover case

Spillovers exist: $Y_i = Y_i(D_i, D_{-i})$

$$
\begin{tikzcd}
 & V_i \arrow[dl, dashed] \arrow[dr, dashed] \arrow[rr, leftrightarrow, dashed] & & V_{-i} \arrow[dl, dashed] \arrow[dr, dashed] & \\
D_i \arrow[rr] & & Y_i & & D_{-i} \arrow[ll]
\end{tikzcd}
$$

Spillover effect and direct effect **vary with unobserved traits** (V_i, V_{-i}):

$$\mathbb{E}[Y_i(d,1) - Y_i(d,0) \mid V_i, V_{-i}], \quad \mathbb{E}[Y_i(1,d) - Y_i(0,d) \mid V_i, V_{-i}]$$

Contribution of This Paper

This paper: a general framework to study heterogeneous treatment effects with spillovers

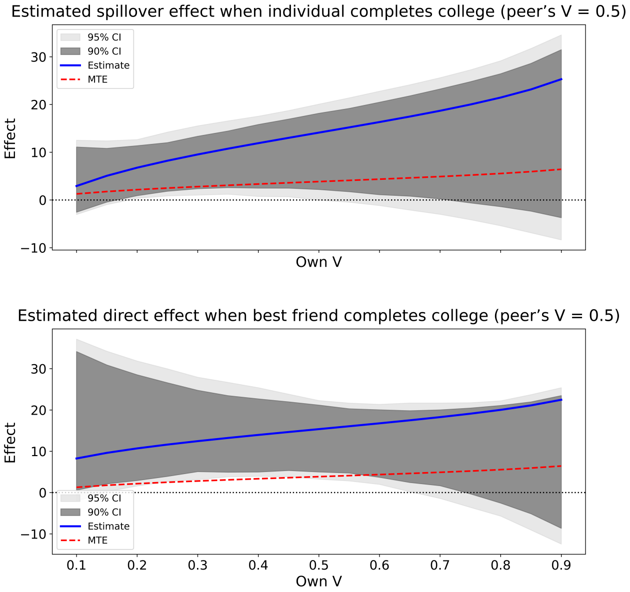
- (V_i, V_{-i}) in a region: Local average treatment effect with spillovers
  - Spillover and direct returns to college for individuals and friends whose college decisions change because of the instrument
- (V_i, V_{-i}) at given values: Marginal treatment effect with spillovers
  - Spillover and direct returns to college that vary continuously with the individual's and the friend's unobserved ability

Contribution: Marginal Treatment Effects With Spillovers

Identify marginal treatment effects with spillovers

- **Point identified nonparametrically** using continuous instruments (e.g., peers' parental background influencing educational choices)
- Can be aggregated to **recover** policy-relevant treatment effects (PRTE) with spillovers
- **Generalize** the standard MTE framework to settings with spillovers

Preview: Application Results



Contribution: Generalized Local Average Effects With Spillover

General identification of local average treatment effects with spillovers

- Applicable to discrete or continuous instruments
- Characterizes the **instrument variation required** for point identification
- With a binary instrument (e.g. cash transfer offer for college completion)
  - Kang & Imbens (2016); Vazquez-Bare (2022); DiTraglia et al. (2023)
  - Rely on one-sided noncompliance
  - E.g., individuals cannot complete college unless receiving the transfer
- **More instrument variation** is needed for less restrictive conditions

Related Literature

- Peer effects with parametric models
  - Manski (1993), Bramoullé, Djebbari, and Fortin (2009), Blume et al. (2015)
- Spillovers under randomized controlled trials
  - Hudgens and Halloran(2008), Aronow and Samii (2017), Vazquez-Bare (2021)
- Spillovers with direct strategic interactions
  - Balat and Han (2023), Hoshino and Yanagi (2023)
  - Require that a unit's treatment does not depend on peer's instruments

Other related literature

Outline

Setting

- Consider i.i.d. groups indexed by g
  - Groups known, predetermined before treatment (e.g., best-friend pairs)
  - Each group contains n units
  - Spillovers exist within groups
- For illustration, consider $n = 2$
  - Unit indexed by $i \in \{0, 1\}$
  - Extendable to $n > 2$

Model and Key Variables

$$Y_{0g} = m_0\left(D_{0g}, D_{1g}, U_{0g}, U_{1g}\right), \quad Y_{1g} = m_1\left(D_{1g}, D_{0g}, U_{1g}, U_{0g}\right)$$
$$D_{0g} = \mathbb{1}\left\{V_{0g} \leq h_0(Z_{0g}, Z_{1g})\right\}, \quad D_{1g} = \mathbb{1}\left\{V_{1g} \leq h_1(Z_{1g}, Z_{0g})\right\}$$

Observe $(Y_{0g}, Y_{1g}, D_{0g}, D_{1g}, Z_{0g}, Z_{1g})$ in each group g

- Outcome $Y_{ig} \in \mathbb{R}$ (e.g. earnings)
- Treatment $D_{ig} \in \{0, 1\}$ (e.g. whether the individual completes college)
  - Extendable to continuous treatments
- Instruments $Z_{ig} \in \mathbb{R}^k$ (e.g. peers' characteristics or cash transfer assignment)

Unobserved variables $(U_{0g}, U_{1g}, V_{0g}, V_{0g})$ in each group g

- Outcome unobservable $U_{ig} \in \mathbb{R}^l$
- Unobserved confounder $V_{ig} \in \mathbb{R}$ (e.g. unobserved ability)

Model: Outcome equation

$$
\begin{aligned}
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\end{aligned}
$$

- Spillovers in outcome: Outcome Y_{ig} depends on peer's treatment $D_{1-i,g}$

Model: Outcome equation

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- Spillovers in outcome: Outcome Y_{ig} depends on peer's treatment $D_{1-i,g}$
- Flexible structure accommodates heterogeneous spillover effects in outcomes
  - **No functional assumptions** on the outcome equations m_0, m_1
  - Outcome Y_{ig} depends on peer's unobservables $U_{1-i,g}$
  - No dimension restrictions on unobservables (U_{0g}, U_{1g})

Model: Outcome equation

$$\begin{aligned} Y_{0g} &= m_0\left(D_{0g}, D_{1g}, U_{0g}, U_{1g}\right), \\ Y_{1g} &= m_1\left(D_{1g}, D_{0g}, U_{1g}, U_{0g}\right) \end{aligned}$$

- Spillovers in outcome: Outcome Y_{ig} depends on peer's treatment $D_{1-i,g}$
- Flexible structure accommodates heterogeneous spillover effects in outcomes
  - **No functional assumptions** on the outcome equations m_0, m_1
  - Outcome Y_{ig} depends on peer's unobservables $U_{1-i,g}$
  - No dimension restrictions on unobservables (U_{0g}, U_{1g})
- Define the potential outcome $Y_{ig}(d, d') \equiv m_i(d, d', U_{ig}, U_{1-i,g})$ ▸ Example: Structural Equations

Model: Treatment equation

$$D_{0g} = \mathbb{1}\left\{V_{0g} \leq h_0(Z_{0g}, Z_{1g})\right\},$$
$$D_{1g} = \mathbb{1}\left\{V_{1g} \leq h_1(Z_{1g}, Z_{0g})\right\}$$

- V_{ig}: continuous unobserved factor driving both treatment and outcomes
  - No distributional restrictions on the joint dependence of V_{0g} and V_{1g}
  - Treatment take-up depends only on V_{ig} (private information)

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- V_{ig}: continuous unobserved factor driving both treatment and outcomes
  - No distributional restrictions on the joint dependence of V_{0g} and V_{1g}
  - Treatment take-up depends only on V_{ig} (private information)

- Unit i's treatment D_{ig} does not depend on peer's treatment $D_{1-i,g}$
  - Balat & Han (2023), Hoshino & Yanagi (2023): allow direct strategic interactions, but $Z_{1-i,g}$ cannot affect D_{ig}

Model: Treatment equation

$$D_{0g} = \mathbb{1}\left\{V_{0g} \leq h_0(Z_{0g}, Z_{1g})\right\},$$
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- D_{ig} can depend on peer's instruments $Z_{1-i,g}$: Spillovers in treatment
  - Accommodates shared or individual-specific instruments: $Z_{0g} = Z_{1g}$ or $Z_{0g} \neq Z_{1g}$

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- Rationalized by a simultaneous incomplete information game (Aradillas-Lopez, 2010)

  Simultaneous incomplete information game
  - Interpret h_0, h_1 as unit 0 and 1's beliefs based on public signals (Z_{0g}, Z_{1g})

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  Simultaneous incomplete information game
  - Interpret h_0, h_1 as unit 0 and 1's beliefs based on public signals (Z_{0g}, Z_{1g})
- **No functional assumptions** on threshold functions h_0, h_1

Monotonicity in Treatment Selection

$D_{ig} = \mathbb{1}\{V_{ig} \leq h_i(Z_{0g}, Z_{1g})\}$ **implies monotonicity** in $D_{ig}(z_0, z_1)$ (cf. Vytlacil, 2002)

- Define *propensity score*: $P_i(z_0, z_1) \equiv \mathbb{P}(D_{ig} = 1 \mid Z_{0g} = z_0, Z_{1g} = z_1), i \in \{0, 1\}$
- $P_i(z_0, z_1)$ identifies threshold function $h_i(z_0, z_1)$
- Observed propensity scores $P_i(z_0, z_1)$ can be ordered
- The order of $P_i(z_0, z_1) \Rightarrow$ the order of $D_{ig}(z_0, z_1)$
- For $Z_{ig} \in \{0, 1\}$:

$$
\begin{aligned}
& P_i(0,0) \leq P_i(0,1) \leq P_i(1,0) \leq P_i(1,1) \\
\Rightarrow \quad & D_{ig}(0,0) \leq D_{ig}(0,1) \leq D_{ig}(1,0) \leq D_{ig}(1,1)
\end{aligned}
$$

Assumptions

1. (Exogeneity) Instruments (Z_{0g}, Z_{1g}) randomly assigned at the group level:

$$\left(Z_{0g}, Z_{1g}\right) \perp\!\!\!\perp \left(V_{0g}, V_{1g}, U_{0g}, U_{1g}\right)$$

2. (Exclusion) (Z_{0g}, Z_{1g}) do not directly affect the outcome $Y_{ig}, i \in \{0, 1\}$:

$$Y_{ig}\left(d_0, d_1, z_0, z_1\right) = Y_{ig}\left(d_0, d_1, z_0', z_1'\right)$$

   for any $z_0 \neq z_0'$ and $z_1 \neq z_1'$

3. (Continuity) V_{ig} is continuously distributed, normalized $Unif(0, 1)$

These assumptions are maintained throughout the talk

Outline

Generalized local average effects

Definition (Generalized local average effects)

i. Generalized local average controlled spillover effects (LACSE):

$$\text{LACSE}_i^{(d)}(P) \equiv \mathbb{E}[Y_i(d,1) - Y_i(d,0) \mid (V_0, V_1) \in P], P \subset (0,1)^2$$

ii. Generalized local average controlled direct effects (LACDE):

$$\text{LACDE}_i^{(d)}(P) \equiv \mathbb{E}\left[Y_i(1,d) - Y_i(0,d) \mid (V_0, V_1) \in P\right], P \subset (0,1)^2$$

Identifying generalized local average effects

Theorem (Identifying generalized local average effects)

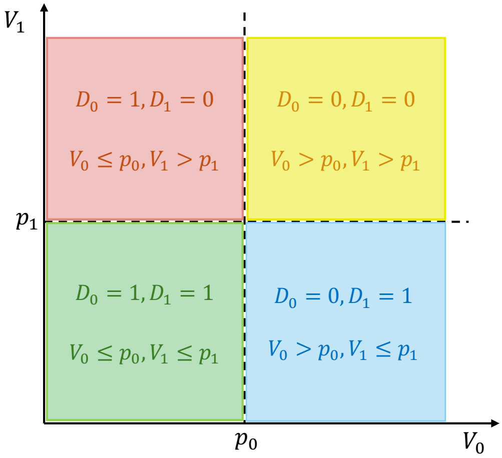
1. If two pairs of propensity scores, (p_0, p_1) and $(p_0, p_1') \in \mathcal{P}$, $p_1' \neq p_1$, exist, LACSE for a specific subpopulation can be identified
2. If two pairs of propensity scores, (p_0, p_1) and $(p_0', p_1) \in \mathcal{P}$, $p_0' \neq p_0$, exist, LACDE for a specific subpopulation can be identified
3. If both conditions in 1 and 2 hold, LACSE and LACDE for a specific subpopulation can be identified

Idea: Relies on variation in the *peer's propensity score* to identify the *spillover effect*, and variation in the *individual's propensity score* to identify the *direct effect*

Mapping Treatment Decisions to Unobserved Heterogeneity

$(p_0, p_1) \in \text{Supp}(P_0, P_1)$: $D_0 = \mathbb{1}\{V_0 \leq p_0\}, D_1 = \mathbb{1}\{V_1 \leq p_1\}$

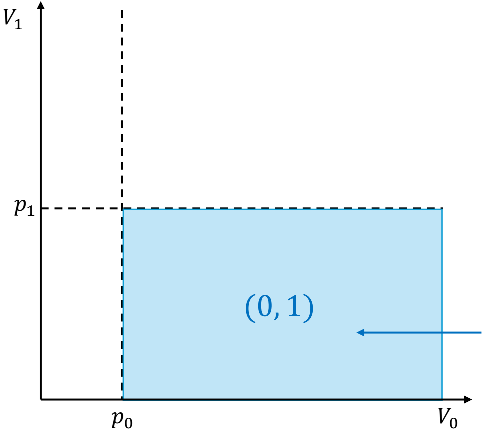
Figure: Treatment Realizations Correspond to Regions in (V_0, V_1)



Local Average of Potential Outcome

Observe $\mathbb{E}[Y_0(1 - D_0)D_1 \mid P_0 = p_0, P_1 = p_1]$

Figure: Local average of potential outcome $Y_0(0, 1)$



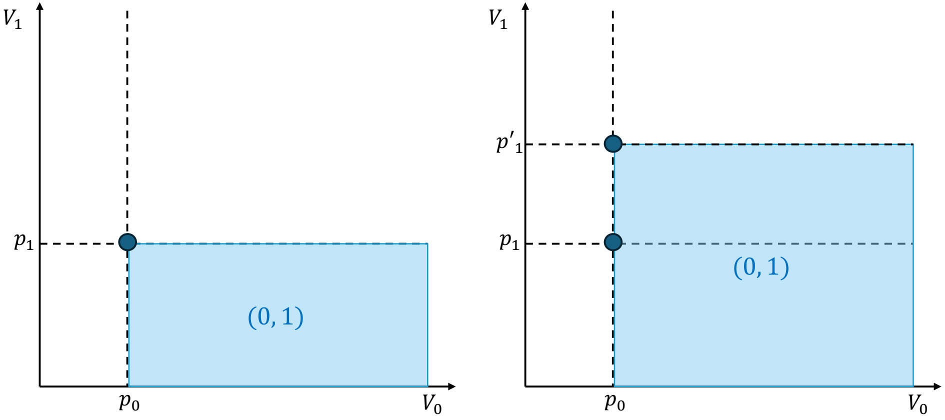
$E[Y_0(1 - D_0)D_1|P_0 = p_0, P_1 = p_1]$

$= E[Y_0(0,1)1\{V_0 > p_0, V_1 \leq p_1\}]$

Propensity Score Variation

Observe (p_0, p_1), (p_0, p_1'), $p_1' > p_1$:

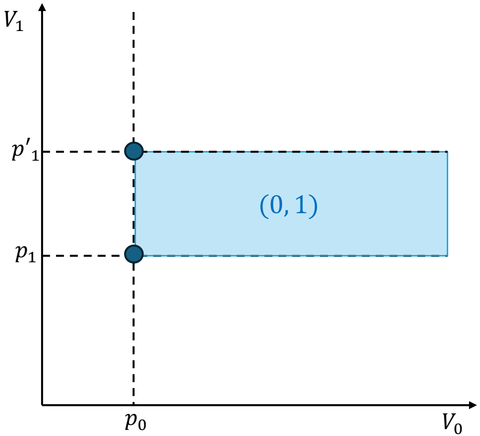
Figure: Local averages of $Y_0(0,1)$ given (p_0, p_1), (p_0, p_1')



Propensity Score Variation

Take the difference between local averages of $Y_0(0,1)$

Figure: Local average of $Y_0(0,1)$ between (p_0, p_1) and (p_0, p_1')

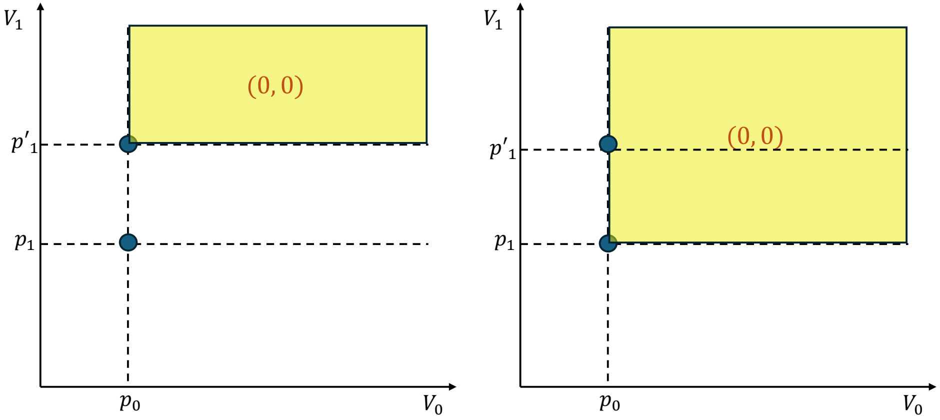


▸ Local average $Y_0(0,0)$

Propensity Score Variation

Observe $(p_0, p_1), (p_0, p_1'), p_1' > p_1$:

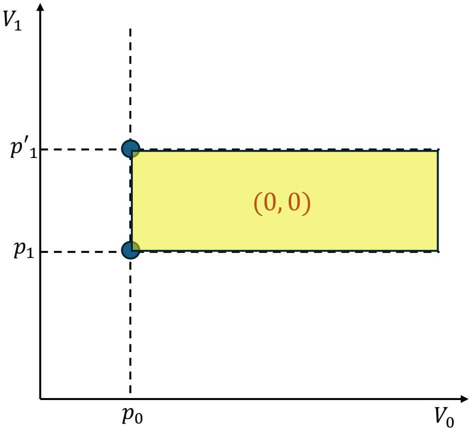
Figure: Local averages of $Y_0(0,0)$ given $(p_0, p_1), (p_0, p_1')$



Propensity Score Variation

Take the difference between local averages of $Y_0(0,0)$

Figure: Local average of $Y_0(0,0)$ between (p_0, p_1) and (p_0, p_1')

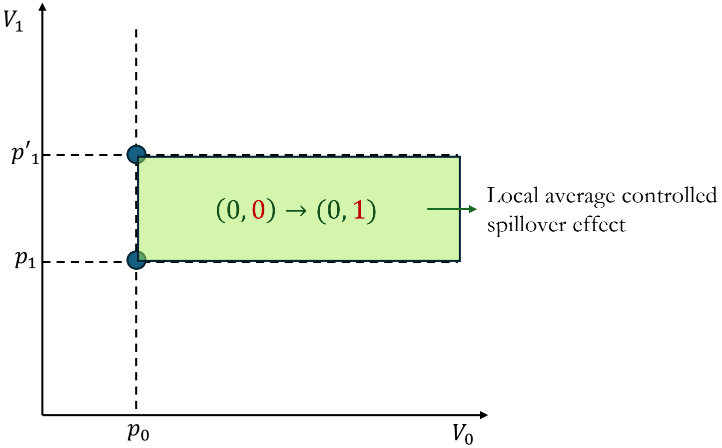


▸ Local average $Y_0(0,1)$

Identify Spillover Effects

Difference between local averages of $Y_0(0,1)$ and $Y_0(0,0)$

Figure: Local average controlled spillover effect between (p_0, p_1) and (p_0, p_1')

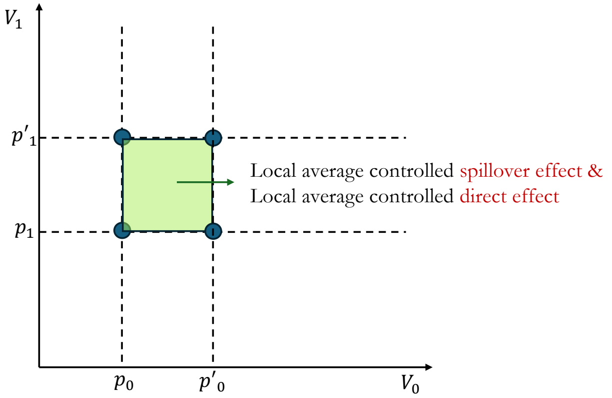


Identify Spillover and Direct Effects

With *"rectangle"* variations (p_0, p_1), (p_0, p'_1), (p'_0, p_1) and (p'_0, p'_1)

- LACSE and LACDE are identified

Figure: LACSE & LACDE between (p_0, p_1), (p_0, p'_1), (p'_0, p_1) and (p'_0, p'_1)



Identify marginal effects

Local Averages With Binary Instrument

- Identification relies on variation in propensity scores
  - Change one unit's propensity score while holding the other's fixed
  - Variation in propensity scores is induced by variation in instruments
- Special case: binary instrument $Z_i \in \{0, 1\}$

	$Z_1 = 0$	$Z_1 = 1$
$Z_0 = 0$	$(P_0(0,0), P_1(0,0))$	$(P_0(0,1), P_1(1,0))$
$Z_0 = 1$	$(P_0(1,0), P_1(0,1))$	$(P_0(1,1), P_1(1,1))$

  - One-sided noncompliance: $P_0(0,0) = P_0(0,1) = 0 \Rightarrow$ local average spillover effect for unit 0
  - Returns to education: Individuals cannot complete college without receiving cash transfer
- Without required variation, point identification with a binary instrument fails

▸ Identification fail with a binary IV

Local Averages Not Point Identified With Binary Instrument

- If the propensity scores lack required variation in the support $\Rightarrow$ **need more variation** in the instruments

- With continuous instrument variation, the previous idea identifies spillover and direct effects over small neighborhood in the interior of propensity score support

- Next: formalize this idea by introducing *marginal contolled spillover/direct effect*

Definition of Marginal Effects

With continuous variation in propensity scores: take limits $p_1' \longrightarrow p_1$ and $p_0' \longrightarrow p_0$

Definition (Marginal effects)

i. Marginal controlled spillover effect (MCSE):

$$\text{MCSE}_i^{(d)}(p_0, p_1) \equiv \mathbb{E}[Y_i(d,1) - Y_i(d,0) \mid V_0 = p_0, V_1 = p_1]$$

ii. Marginal controlled direct effect (MCDE):

$$\text{MCDE}_i^{(d)}(p_0, p_1) \equiv \mathbb{E}[Y_i(1,d) - Y_i(0,d) \mid V_0 = p_0, V_1 = p_1]$$

Definition of Marginal Effects

With continuous variation in propensity scores: take limits $p_1' \to p_1$ and $p_0' \to p_0$

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ii. Marginal controlled direct effect (MCDE):

$$\text{MCDE}_i^{(d)}(p_0, p_1) \equiv \mathbb{E}[Y_i(1,d) - Y_i(0,d) \mid V_0 = p_0, V_1 = p_1]$$

- Define the copula between V_0 and V_1 as

$$C(p_0, p_1) \equiv \mathbb{P}(V_0 \leq p_0, V_1 \leq p_1)$$

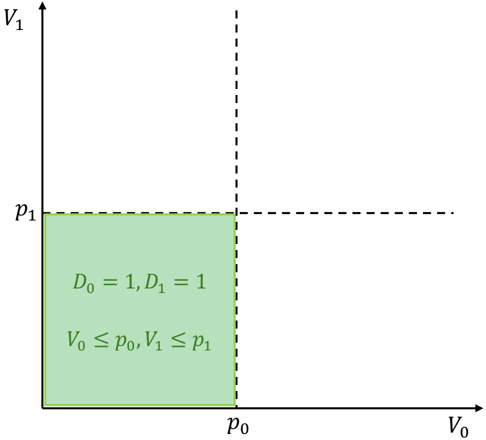
- Define marginal treatment response (MTR) function

$$m_i^{(d_0,d_1)}(p_0, p_1) \equiv \mathbb{E}[Y_i(d_0, d_1) \mid V_0 = p_0, V_1 = p_1]$$

Identifying Copula

Lemma: $\mathbb{P}(D_0 = 1, D_1 = 1 \mid P_0 = p_0, P_1 = p_1)$ identifies $C(p_0, p_1)$, $(p_0, p_1) \in \text{Supp}(P_0, P_1)$

Figure: Identify joint distribution of (V_0, V_1)



Identifying Copula Density

Lemma: $\mathbb{P}(D_0 = 1, D_1 = 1 \mid P_0 = p_0, P_1 = p_1)$ identifies $C(p_0, p_1)$, $(p_0, p_1) \in \text{Supp}(P_0, P_1)$

Assumption 4: (Continuous instruments) At least one component of (Z_0, Z_1) is continuous

- Taking cross derivative of $C(p_0, p_1)$ to identify copula density $c_{V_0,V_1}(p_0, p_1)$

$$\frac{\partial^2 \mathbb{E}\left[D_0 D_1 \mid P_0 = p_0, P_1 = p_1\right]}{\partial p_0 \partial p_1} = c_{V_0,V_1}(p_0, p_1)$$

  if $C(\cdot, \cdot)$ is twice differentiable

Identifying Marginal Controlled Effects

Theorem (Identifying marginal controlled effects)

The marginal controlled spillover effects (MCSEs) are identified as

$$\operatorname{sgn}(2d-1) \cdot \frac{\partial^2 \mathbb{E}\left[Y_i \mathbb{1}\left\{D_i = d\right\} \mid P_0 = p_0, P_1 = p_1\right]}{\partial p_0 \partial p_1} \Bigg/ \frac{\partial^2 \mathbb{E}\left[D_0 D_1 \mid P_0 = p_0, P_1 = p_1\right]}{\partial p_0 \partial p_1}$$

The marginal controlled direct effects (MCDEs) are identified as

$$\operatorname{sgn}(2d-1) \cdot \frac{\partial^2 \mathbb{E}\left[Y_i \mathbb{1}\left\{D_{1-i} = d\right\} \mid P_0 = p_0, P_1 = p_1\right]}{\partial p_0 \partial p_1} \Bigg/ \frac{\partial^2 \mathbb{E}\left[D_0 D_1 \mid P_0 = p_0, P_1 = p_1\right]}{\partial p_0 \partial p_1}$$

for $d \in \{0, 1\}$ and (p_0, p_1) in the interior of $\operatorname{Supp}(P_0, P_1)$ ▸ Twice difference strategy ▸ Proof sketch

Identifying Marginal Controlled Effects

Theorem (Identifying marginal controlled effects)

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$$\operatorname{sgn}(2d-1)\cdot \frac{\partial^2 \mathbb{E}\left[Y_i \mathbb{1}\left\{D_{1-i} = d\right\} \mid P_0 = p_0, P_1 = p_1\right]}{\partial p_0 \partial p_1} \Bigg/ \frac{\partial^2 \mathbb{E}\left[D_0 D_1 \mid P_0 = p_0, P_1 = p_1\right]}{\partial p_0 \partial p_1}$$

for $d \in \{0,1\}$ and (p_0, p_1) in the interior of $\operatorname{Supp}(P_0, P_1)$ Twice difference strategy Proof sketch

Extension

- Identify MCSE & MCDE with *discrete instruments* by imposing parametric assumptions (Brinch et al., 2017), or apply methods similar to Mogstad et al. (2018)
- Identification when groups differ in size Extension: Exposure mapping

Policy Relevant Treatment Effects

Many PRTEs are identified as integrals of MCSE and MCDE

- LACSE: Units with $h_i(z_0, z_1) < V_i \leq h_i(z_1, z_0)$ correspond to complier

$$\mathbb{E}\left[Y_i(d,1) - Y_i(d,0) \mid T_i = c, T_{-i} = c\right]$$
$$= \frac{1}{\mathbb{P}(T_0 = c, T_1 = c)} \int_{h_1(z_0,z_1)}^{h_1(z_1,z_0)} \int_{h_0(z_0,z_1)}^{h_0(z_1,z_0)} \text{MCSE}_i(d_0; v_0, v_1)\, c_{V_0,V_1}(v_0, v_1)\, dv_0 dv_1,$$
$$\mathbb{P}(T_0 = c, T_1 = c) = \int_{h_1(z_0,z_1)}^{h_1(z_1,z_0)} \int_{h_0(z_0,z_1)}^{h_0(z_1,z_0)} c_{V_0,V_1}(v_0, v_1)\, dv_0 dv_1$$

- Additional PRTE results: see paper ▸ Examples

Connection to Standard Marginal Treatment Effects

- Without spillovers
  - Marginal controlled **spillover effect** = 0
  - Marginal controlled **direct effect** $\Rightarrow$ **standard MTE:** $\mathbb{E}[Y_i(1) - Y_i(0) \mid V_i]$

▸ Marginal effects without spillovers

- With spillovers standard MTE may lose causal interpretation
  - MTE estimand identifies averaged MCDEs **plus residual**
  - Residuals disappear only if:
    - D_i does not depend on peer's instrument, and
    - Instruments Z_{0g} and Z_{1g} are independent within group

▸ Marginal effects with spillovers ▸ Testable implications

Outline

Parametric Estimation Procedure

Semiparametric results in paper: nonparametric convergence rate ▸ Semiparametric estimation

Use parametric estimation when sample size is limited

Parametric Assumptions

1. $D_i = \mathbb{1}\{\widetilde{V}_i \leq h_i(Z_i, Z_{-i}; \theta_i)\}$, h_i is a K-th order polynomial, $\widetilde{V}_i \sim N(0,1)$
2. C_{V_0,V_1} is given by Gaussian copula with correlation ρ, $V_i = \Phi(\widetilde{V}_i)$
3. The marginal treatment response function satisfies

$$\mathbb{E}\left[Y_i\left(d, d'\right) \mid V_0 = v_0, V_1 = v_1\right] = \alpha_{idd',0} + \alpha_{idd',1}\Phi^{-1}\left(v_0\right)$$
$$+\alpha_{idd',2}\Phi^{-1}\left(v_1\right) + \alpha_{idd',3}\Phi^{-1}\left(v_0\right)\Phi^{-1}\left(v_1\right)$$

- Point identify MCSE & MCDE with discrete instruments (cf. Brinch et al., 2017)

Parametric Estimation Procedure

First stage and second stage: Estimate polynomial parameters θ_i and correlation ρ via maximum likelihood

Parametric Estimation Procedure

First stage and second stage: Estimate polynomial parameters θ_i and correlation ρ via maximum likelihood

Third stage: Estimate outcome parameters $\alpha_{idd'}$ through separate regressions

$$
\begin{aligned}
&\mathbb{E}[Y_i \mathbb{1}\{D_i = d, D_{-i} = d'\} \mid P_{0g} = p_0, P_{1g} = p_1] \\
=&\alpha_{idd',0} I^0_{dd'}(p_0, p_1, \rho) + \alpha_{idd',1} I^1_{dd'}(p_0, p_1, \rho) + \alpha_{idd',2} I^2_{dd'}(p_0, p_1, \rho) + \alpha_{idd',3} I^3_{dd'}(p_0, p_1, \rho)
\end{aligned}
$$

- $I_{dd'}$ terms are integrals tied to first- and second-stage estimands
- Generated by plug-in estimators and Gauss-Hermite quadrature (fast, accurate numerical integration)

Parametric Estimation Procedure

First stage and second stage: Estimate polynomial parameters θ_i and correlation ρ via maximum likelihood

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$$\begin{aligned}&\mathbb{E}[Y_i \mathbb{1}\{D_i = d, D_{-i} = d'\} \mid P_{0g} = p_0, P_{1g} = p_1] \\ =&\alpha_{idd',0} I^0_{dd'}(p_0, p_1, \rho) + \alpha_{idd',1} I^1_{dd'}(p_0, p_1, \rho) + \alpha_{idd',2} I^2_{dd'}(p_0, p_1, \rho) + \alpha_{idd',3} I^3_{dd'}(p_0, p_1, \rho)\end{aligned}$$

- $I_{dd'}$ terms are integrals tied to first- and second-stage estimands
- Generated by plug-in estimators and Gauss-Hermite quadrature (fast, accurate numerical integration)

Inference: Nonparametric bootstrap

- Monte Carlo simulation shows consistency and correct coverage Simulation results

Outline

Application: Best-Friend Spillovers in Education Returns

Direct and spillover effects of returns to education within best-friend groups using Add Health data

- Best-friend pair: mutual best-friend nominations in high school
- Outcome Y: Log of total personal yearly pre-tax income
- Treatment D: 1 if completed ≥ 16 years of education, 0 otherwise
- Instrument Z: Average parental education level of the individual's non-best friends
  - Peers' parental backgrounds can influence college completion through self-confidence or aspirations (Cools et al., 2022)
- Covariates X: Age, gender, race, health status, and family income
- The sample comprises 1,019 best-friend pairs: apply parametric procedure

Application: Assumptions

- Best friends' educational choices do not directly influence each other
  - Decision depends on private education costs

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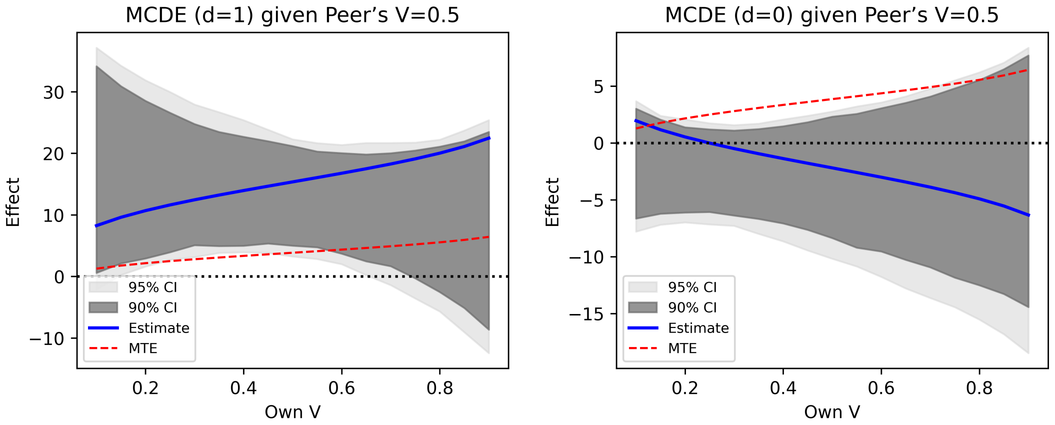
- Family background of non-best friends is independent
  - Reflects weaker social ties
  - Reasonably independent after controlling for covariates

Application: Assumptions

- Best friends' educational choices do not directly influence each other
  - Decision depends on private education costs
- Family background of non-best friends is independent
  - Reflects weaker social ties
  - Reasonably independent after controlling for covariates
- Family background of non-best friends does not affect outcomes
  - Weaker social ties unlikely to shape long-term labor market outcomes

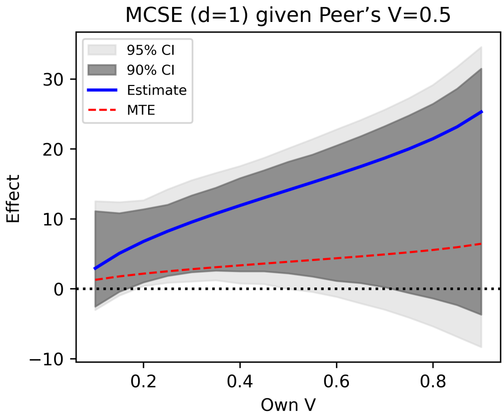
Application: Results

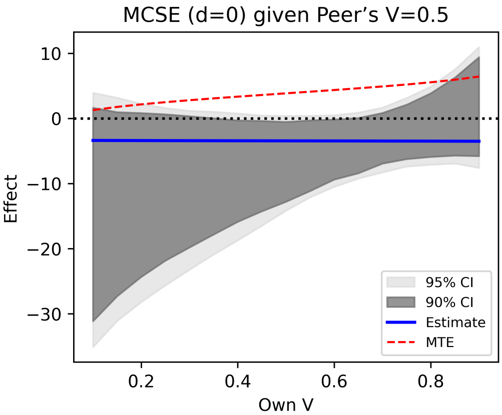
Positive dependence: Correlation between best friends' unobservables V_i and V_{-i} is 0.36



Parametric estimates of MCDE with 90% CIs (dark gray areas) and 95% CIs (light gray areas)

Application: Results





Parametric estimates of MCSE with 90% CIs (dark gray areas) and 95% CIs (light gray areas)

Conclusion

- Enable identification and estimation of heterogeneous direct and spillover effects
- Consider local average controlled spillover and direct effects
- Define and identify marginal controlled spillover and direct effects
- Provide semiparametric and parametric estimation and apply to best-friend college returns in Add Health
- Several extensions are developed in the paper
  - Identify MCSE & MCDE with discrete instruments
  - Identify MCSE & MCDE with continuous treatment
  - Identify MCSE & MCDE for groups of varying sizes

Thank You!

I welcome your questions

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Outline

Positioning in the Literature: Multivalued Treatments

View the group-level treatment vector as a multivalued treatment

- The spillover setting is similar to the multivalued treatment framework in Lee and Salanié (2018)
- Lee and Salanié (2018) require an additional exclusion restriction on instruments
  - Translated to spillover model: requires unit's treatment not to depend on peer's instruments
- Marginal effects are point identified without extra exclusion restriction in the spillover model
- The two frameworks are not nested

▸ Back to Literature

Treatment response and structural functions

$Y_{ig}(d,d')$ generally serve as the reduced form of structural models with endogenous effects

- $Y_{ig}(d,d')$ is linear when structural functions are linear in treatments and outcomes

$$
\begin{aligned}
Y_{0g} =& \alpha_0 + \alpha_1 D_{0g} + \alpha_2 D_{1g} + \alpha_3 Y_{1g} + U_{0g} + \gamma_1 U_{1g}, \\
Y_{1g} =& \beta_0 + \beta_1 D_{1g} + \beta_2 D_{0g} + \beta_3 Y_{0g} + U_{1g} + \gamma_2 U_{0g} \\
\implies Y_{0g} =& \frac{\alpha_0 + \alpha_3\beta_0 + (\alpha_1 + \alpha_3\beta_2)D_{0g} + (\alpha_2 + \alpha_3\beta_1)D_{1g}}{1 - \alpha_3\beta_3} \\
& + \frac{(1 + \alpha_3\gamma_2)U_{0g} + (\gamma_1 + \alpha_3)U_{1g}}{1 - \alpha_3\beta_3}, \\
Y_{1g} =& \frac{\beta_0 + \beta_3\alpha_0 + (\beta_1 + \beta_3\alpha_2)D_{1g} + (\beta_2 + \beta_3\alpha_1)D_{0g}}{1 - \alpha_3\beta_3} \\
& + \frac{(1 + \beta_3\gamma_1)U_{1g} + (\beta_3 + \gamma_2)U_{0g}}{1 - \alpha_3\beta_3}, \alpha_3\beta_3 \neq 1
\end{aligned}
$$

- Relations between treatment response and structural functions are unclear when structural functions are nonlinear Back to outcome model

Simultaneous game with incomplete information

		Player 1	
		$D_1 = 1$	$D_1 = 0$
Player 0	$D_0 = 1$	$-V_0 + \alpha_0$, $-V_1 + \alpha_1$	$-V_0, 0$
	$D_0 = 0$	$0, -V_1$	$0,\ 0$

- Information structure and beliefs
  - V_i is only privately observed by player i
  - Z is a publicly observed vector of signals
  - Each player forms a subjective belief $\Pr(D_0, D_1 \mid Z)$
- Optimal decisions

$$D_0 = \mathbb{1}\{V_0 \leq \alpha_0 \underbrace{\Pr_0(D_1 = 1 \mid D_0 = 1, Z)}_{\text{Player 0's belief, function of } Z}\}$$

$$D_1 = \mathbb{1}\{V_1 \leq \alpha_1 \underbrace{\Pr_1(D_0 = 1 \mid D_1 = 1, Z)}_{\text{Player 1's belief, function of } Z}\}.$$

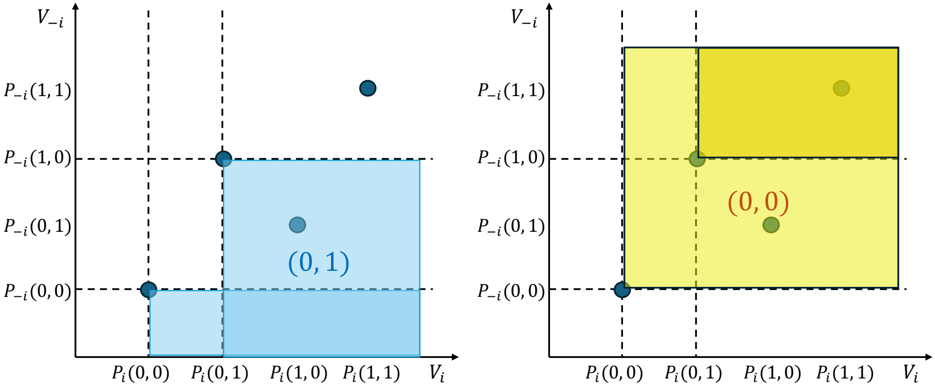
  - Aradillas-Lopez (2010) gives conditions for the existence and uniqueness of equilibrium beliefs Back to treatment model

Local Averages Not Point Identified With Binary Instrument

Suppose that $P_i(1,1) > P_i(1,0) > P_i(0,1) > P_i(0,0)$ (monotonicity)

- **Cannot point identify** local averages of different potential outcomes for same subpopulation

Figure: Local averages without one-sided noncompliance



Back to Identification with binary IV

Identifying decision threshold

$$D_{ig} = \mathbb{1}\left\{V_{ig} \leq h(Z_{ig}, Z_{-ig})\right\}$$

- Identifying $h(\cdot, \cdot)$
  - Propensity score: $P_i(z_1, z_2) \equiv \mathbb{P}(D_i = 1 \mid Z_i = z_1, Z_{-i} = z_2)$
  - $P_i(z_1, z_2)$ identifies $h(z_1, z_2)$ under exogeneity and continuity

$$\begin{aligned} P_i(z_1, z_2) &= \mathbb{P}\left(V_i \leq h\left(z_1, z_2\right) \mid Z_i = z_1, Z_{-i} = z_2\right) \\ &= \mathbb{P}\left(V_i \leq h\left(z_1, z_2\right)\right) \\ &= h\left(z_1, z_2\right) \end{aligned}$$

Identifying copula density of unobservables

$$D_{ig} = \mathbb{1}\left\{V_{ig} \leq h(Z_{ig}, Z_{-ig})\right\}$$

- Identifying copula density $f_{V_i,V_{-i}}(\cdot,\cdot)$
  - Define $C(p_1, p_2) \equiv \mathbb{P}(D_i = 1, D_{-i} = 1 \mid P_i = p_1, P_{-i} = p_2)$
  - $C(p_1, p_2)$ identifies the copula of V_i and V_{-i}

    $$C(p_1, p_2) = \mathbb{P}\left(V_i \leq p_1, V_{-i} \leq p_2\right)$$

  - Taking cross derivative of $C(p_1, p_2)$ to identify $f_{V_i,V_{-i}}(p_1, p_2)$

    $$\frac{\partial^2 C(p_1, p_2)}{\partial p_2 \partial p_1} = f_{V_i,V_{-i}}(p_1, p_2)$$

    if $C(\cdot,\cdot)$ is twice differentiable
  - The derivative requires continuous instruments

Identifying marginal treatment response

$$Y_{ig} = \sum_{d,d' \in \{0,1\}} Y_i(d,d') \mathbb{1}\{D_i = d\} \mathbb{1}\{D_i = d'\}$$

- Define marginal treatment response function $m_i^{(d_1,d_2)}(p_1, p_2) \equiv \mathbb{E}[Y_i(d_1, d_2) \mid V_i = p_1, V_{-i} = p_2]$

$$\begin{aligned}
&\mathbb{E}\left[Y_i D_i D_{-i} \mid P_i = p_1, P_{-i} = p_2\right] \\
=& \int_0^{p_2} \int_0^{p_1} \left\{ m_i^{(1,1)}(v_1, v_2) \cdot f_{V_i, V_{-i}}(v_1, v_2) \right\} dv_1 dv_2
\end{aligned}$$

  if $m_i^{(1,1)}(\cdot, \cdot)$ is continuous

- Taking cross derivative of $\mathbb{E}[Y_i D_i D_{-i} \mid P_i = p_1, P_{-i} = p_2]$ to identify $m_i^{(1,1)}(p_1, p_2)$

$$\begin{aligned}
&\frac{\partial^2 \mathbb{E}\left[Y_i D_i D_{-i} \mid P_i = p_1, P_{-i} = p_2\right]}{\partial p_2 \partial p_1} \frac{1}{f_{V_i, V_{-i}}(p_1, p_2)} \\
=& m_i^{(1,1)}(p_1, p_2)
\end{aligned}$$

  if $\mathbb{E}[Y_i D_i D_{-i} \mid \cdot, \cdot]$ is twice differentiable Main theorem

Policy relevant treatment effect

Propensity score under policy a: $P_i^a\left(Z_i^a, Z_{1-i}^a\right) = \mathbb{P}\left(D_i^a = 1 \mid Z_i^a, Z_{1-i}^a\right)$

- Assumption (Policy invariances): Distribution of $\left\{\left(U_0^a, U_1^a, V_0^a, V_1^a\right)\right\}_{d,d' \in \{0,1\}}$ is invariant with a
- Two policies a, a' such that $P_i^{a'} = P_i^a + \varepsilon$, $\varepsilon > 0$
- Policy relevant treatment effect is $\mathbb{E}\left[Y_i^{a'} - Y_i^a\right] / \Delta P$,

$$
\begin{aligned}
\mathbb{E}\left[Y_i^{a'} - Y_i^a\right] = \int_0^1 \int_0^1 \Big\{ & \mathrm{MCDE_i}(0; \mathrm{p}_0, \mathrm{p}_1)\, \mathbb{P}\left(p_0 - \varepsilon \leq P_i^a \leq p_0, P_{-i}^a \leq p_1 - \varepsilon\right) \\
& + \mathrm{MCSE}_i(0; p_0, p_1)\mathbb{P}\left(P_i^a \leq p_0 - \varepsilon, p_1 - \varepsilon \leq P_{-i}^a < p_1\right) \\
& + \mathrm{MCDE}_i(1; p_0, p_1)\mathbb{P}\left(p_0 - \varepsilon \leq P_i^a \leq p_0, p_1 \leq P_{-i}^a\right) \\
& + \mathrm{MCSE}_i(1; p_0, p_1)\mathbb{P}\left(p_0 \leq P_i^a, p_1 - \varepsilon \leq P_{-i}^a < p_1\right) \\
& + \left(\mathrm{MCDE}_i(1; p_0, p_1) + \mathrm{MCSE}_i(0; p_0, p_1)\right) \\
& \quad \mathbb{P}\left(p_0 - \varepsilon \leq P_i^a < p_0, p_1 - \varepsilon \leq P_{-i}^a < p_1\right) \Big\} c_{V_i, V_{-i}}(p_0, p_1)\, dp_0\, dp_1
\end{aligned}
$$

Back to PRTE

Connection with local average effects

- In the spillover setting, MCSE and MCDE can recover local average effects, but the reverse is not true
- Vazquez-Bare (2022) considers a similar setting with binary instrument $Z_i \in \{z_0, z_1\}$
  - Monotonicity: $D_i(z_1, z_1) \geq D_i(z_1, z_0) \geq D_i(z_0, z_1) \geq D_i(z_0, z_0)$
  - Define population types

$D_i(1,1)$	$D_i(1,0)$	$D_i(0,1)$	$D_i(0,0)$	Type (T_i)
1	1	1	1	always-taker(at)
1	1	1	0	social-interaction complier (sc)
1	1	0	0	complier (c)
1	0	0	0	group complier (gc)
0	0	0	0	never-taker (nt)

- **Partially identify** type proportions and local average direct/spillover effects
  - Hard to identify marginal effects by taking derivatives

Connection with local average effects

- Once identifying copula density and marginal effects
  - Choose z_0, z_1 such that $h_i(z_0, z_0) \leq h_i(z_0, z_1) \leq h_i(z_1, z_0) \leq h_i(z_1, z_1)$
- Create a mapping from unobservable V_i to type T_i, e.g.,

$$T_i = c \text{ if } h_i(z_0, z_1) < V_i \leq h_i(z_1, z_0)$$

- Identify the type proportions and relevant local average effects
  - Probability that both units are compliers

$$\mathbb{P}(T_i = c, T_{-i} = c) = \int_{h_{-i}(z_0,z_1)}^{h_{-i}(z_1,z_0)} \int_{h_i(z_0,z_1)}^{h_i(z_1,z_0)} f_{V_i,V_{-i}}(v_0, v_1)\, dv_0 dv_1$$

  - Local average spillover effects given both units are complier

$$\mathbb{E}[Y_i(d_0, 1) - Y_i(d_0, 0) \mid T_i = c, T_{-i} = c]\mathbb{P}(T_i = c, T_{-i} = c)$$
$$= \int_{h_{-i}(z_0,z_1)}^{h_{-i}(z_1,z_0)} \int_{h_i(z_0,z_1)}^{h_i(z_1,z_0)} \mathrm{MCSE}_i(d_0; v_0, v_1) f_{V_i,V_{-i}}(v_0, v_1)\, dv_0 dv_1,$$

Back to comparisons

Marginal Effects Without Spillovers

Without spillovers, marginal controlled effects are the same as standard marginal treatment effect

- Suppose that $Y_i(D_i, d) = Y_i(D_i, d') \equiv Y_i(D_i)$, $h_i(Z_i, z) = h_i(Z_i, z') \equiv h_i(Z_i)$

Marginal Effects Without Spillovers

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- Suppose that $Y_i(D_i, d) = Y_i(D_i, d') \equiv Y_i(D_i)$, $h_i(Z_i, z) = h_i(Z_i, z') \equiv h_i(Z_i)$
- The propensity score identifies

$$P_i(z_0, z_1) = h_i(z_0)$$

Marginal Effects Without Spillovers

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- The propensity score identifies

$$P_i(z_0, z_1) = h_i(z_0)$$

- The differentiation of $C_g(\cdot, \cdot)$ is

$$\partial^2 C_g\left(p_1, p_2\right) / \partial p_2 \partial p_1 = 1$$

Marginal Effects Without Spillovers

Without spillovers, marginal controlled effects are the same as standard marginal treatment effect

- Suppose that $Y_i(D_i, d) = Y_i(D_i, d') \equiv Y_i(D_i)$, $h_i(Z_i, z) = h_i(Z_i, z') \equiv h_i(Z_i)$
- The propensity score identifies

$$P_i(z_0, z_1) = h_i(z_0)$$

- The differentiation of $C_g(\cdot, \cdot)$ is

$$\partial^2 C_g\left(p_1, p_2\right) / \partial p_2 \partial p_1 = 1$$

- Marginal controlled **spillover effects** are identified as **0**
- Marginal controlled **direct effects** are identified as **standard MTE:** $\mathbb{E}[Y_i(1) - Y_i(0) \mid V_i]$

Back to Connection to MTE

Comparison to Standard MTE

Standard MTE may lose causal interpretation if spillovers exist

- $\partial \mathbb{E}\left[Y_i \mid P_i\left(Z_i\right)=p_0\right] / \partial p_0$ identifies averaged MCDEs **plus some residuals**

$$\begin{aligned}
& \int_0^1 \int_0^{p_1} \operatorname{MCDE}_i\left(1 ; p_1, p_2\right) f_{V_i, V_{-i}}\left(p_0, v_1\right) f_{P_{-i} \mid P_i=p_0}\left(p_1\right) d v_1 d p_1 \\
+ & \int_0^1 \int_{p_1}^1 \operatorname{MCDE}_i\left(0 ; p_1, p_2\right) f_{V_i, V_{-i}}\left(p_0, v_1\right) f_{P_{-i} \mid P_i=p_0}\left(p_1\right) d v_1 d p_1 \\
+ & \text{ residuals }
\end{aligned}$$

Comparison to Standard MTE

Standard MTE may lose causal interpretation if spillovers exist

- $\partial \mathbb{E}\left[Y_i \mid P_i\left(Z_i\right)=p_0\right] / \partial p_0$ identifies averaged MCDEs **plus some residuals**

$$
\begin{aligned}
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+ & \int_0^1 \int_{p_1}^1 \mathrm{MCDE}_i\left(0 ; p_1, p_2\right) f_{V_i, V_{-i}}\left(p_0, v_1\right) f_{P_{-i} \mid P_i=p_0}\left(p_1\right) d v_1 d p_1 \\
+ & \text{residuals}
\end{aligned}
$$

- Residuals are generally nonzero
- Residuals are zero when both conditions hold:
  - A unit's treatment decision does not depend on peer's instrument
  - The instruments Z_{0g} and Z_{1g} are mutually independent within groups

Back to Connection to MTE

Testable Implications

Nesting inequality: Copula density and probabilities are nonnegative $\implies$

$$\frac{\partial^2}{\partial p_1 \partial p_0}\mathbb{E}\left[\mathbb{1}\left\{Y_i \in A_1, Y_{-i} \in A_2\right\}\mathbb{1}\left\{D_i = d, D_{-i} = d\right\} \mid P_i = p_0, P_{-i} = p_1\right]$$
$$= \mathbb{P}\left(Y_i\left(d,d\right) \in A_1, Y_{-i}\left(d,d\right) \in A_2 \mid V_i = p_0, V_{-i} = p_1\right) c_{V_i,V_{-i}}(p_0,p_1) \geq 0,$$
$$- \frac{\partial^2}{\partial p_1 \partial p_0}\mathbb{E}\left[\mathbb{1}\left\{Y_i \in A_1, Y_{-i} \in A_2\right\}\mathbb{1}\left\{D_i = d, D_{-i} = 1-d\right\} \mid P_i = p_0, P_{-i} = p_1\right]$$
$$= \mathbb{P}\left(Y_i\left(d,1-d\right) \in A_1, Y_{-i}\left(d,1-d\right) \in A_2 \mid V_i = p_0, V_{-i} = p_1\right) c_{V_i,V_{-i}}(p_0,p_1) \geq 0$$

Testable Implications

Nesting inequality: Copula density and probabilities are nonnegative $\Longrightarrow$

$$
\begin{aligned}
&\frac{\partial^2}{\partial p_1 \partial p_0} \mathbb{E}\left[\mathbb{1}\left\{Y_i \in A_1, Y_{-i} \in A_2\right\} \mathbb{1}\left\{D_i = d, D_{-i} = d\right\} \mid P_i = p_0, P_{-i} = p_1\right] \\
&= \mathbb{P}\left(Y_i(d,d) \in A_1, Y_{-i}(d,d) \in A_2 \mid V_i = p_0, V_{-i} = p_1\right) c_{V_i,V_{-i}}(p_0,p_1) \geq 0, \\
&- \frac{\partial^2}{\partial p_1 \partial p_0} \mathbb{E}\left[\mathbb{1}\left\{Y_i \in A_1, Y_{-i} \in A_2\right\} \mathbb{1}\left\{D_i = d, D_{-i} = 1-d\right\} \mid P_i = p_0, P_{-i} = p_1\right] \\
&= \mathbb{P}\left(Y_i(d,1-d) \in A_1, Y_{-i}(d,1-d) \in A_2 \mid V_i = p_0, V_{-i} = p_1\right) c_{V_i,V_{-i}}(p_0,p_1) \geq 0
\end{aligned}
$$

Index sufficiency: For any $(z_0, z_1) \neq (\tilde{z}_0, \tilde{z}_1)$ such that $P_i(z_0, z_1) = P_i(\tilde{z}_0, \tilde{z}_1)$, $P_{-i}(z_0, z_1) = P_{-i}(\tilde{z}_0, \tilde{z}_1)$,

$$
\begin{aligned}
&\mathbb{E}\left[\mathbb{1}\left\{Y_i \in A_1, Y_{-i} \in A_2\right\} \mathbb{1}\left\{D_i = d, D_{-i} = d'\right\} \mid Z_i = z_0, Z_{-i} = z_1\right] \\
=&\mathbb{E}\left[\mathbb{1}\left\{Y_i \in A_1, Y_{-i} \in A_2\right\} \mathbb{1}\left\{D_i = d, D_{-i} = d'\right\} \mid Z_i = \tilde{z}_0, Z_{-i} = \tilde{z}_1\right]
\end{aligned}
$$

Extension: (i) prove sharpness; (ii) develop implementation Back to Connection to MTE

Semiparametric Estimation Procedure

The data $\{(Y_{0g}, Y_{1g}, D_{0g}, D_{1g}, Z_{0g}, Z_{1g}) : g = 1, \cdots, G\}$ is i.i.d.

Estimand:

$$m_i^{(d,d')}(p_0, p_1) = \frac{\partial^2 \mathbb{E}\left[Y_{idd'} \mid P_0 = p_0, P_1 = p_1\right]}{\partial p_0 \partial p_1} \Big/ \frac{\partial^2 \mathbb{E}\left[D_0 D_1 \mid P_0 = p_0, P_1 = p_1\right]}{\partial p_0 \partial p_1},$$
$$Y_{idd'} \equiv Y_i \mathbb{1}\{D_0 = d, D_1 = d'\}, i, d, d' \in \{0, 1\}$$

- Semiparametric estimation methods (Carneiro and Lee, 2009)

First stage: Estimate $\mathbb{P}(D_i = 1 \mid Z_0 = z_0, Z_1 = z_1)$ using partial linear regression and series estimation (Belloni et al., 2015)

Second stage: Estimate $m_i^{(d,d')}(p_0, p_1)$ using two local polynomial regressions ▸ Detailed procedure

Asymptotic distributions

Theorem: Under regularity conditions,

$$(Gh_G^6)^{1/2}\left(\hat{m}_i^{(d,d')}(p_0,p_1) - m_i^{(d,d')}(p_0,p_1)\right) \xrightarrow{d} N\left(0, V_{dd'}(p_0,p_1)\right)$$

- Limiting distribution of MCSEs & MCDEs can be characterized

$$(Gh_G^6)^{1/2}\left(\widehat{MCSE}_i(d;p_0,p_1) - MCSE_i(d;p_0,p_1)\right) \xrightarrow{d} N\left(0, V_{d1}(p_0,p_1) + V_{d0}(p_0,p_1)\right),$$

$$\widehat{MCSE}_i(d;p_0,p_1) = \hat{m}_i^{(d,1)}(p_0,p_1) - \hat{m}_i^{(d,0)}(p_0,p_1),$$

$$MCSE_i(d;p_0,p_1) = m_i^{(d,1)}(p_0,p_1) - m_i^{(d,0)}(p_0,p_1)$$

Back to parametric estimation

First stage: Estimate propensity score

- Assume $\mathbb{P}(D_i = 1 \mid Z_0 = z_0, Z_1 = z_1) \equiv P_i(z_0, z_1)$ is partially linear

$$P_i(z_0, z_1) = \varphi_{01}(z_{01}) + \cdots + \varphi_{0d}(z_{0d}) + \varphi_{11}(z_{11}) + \cdots + \varphi_{1d}(z_{1d})$$

- Approximate φ_{ij} via a spline basis $\{p_k : k = 1, 2, \ldots\}$
- Given a positive integer κ, define regressors

$$P_\kappa(z_0, z_1) = [p_1(z_{01}), \cdots, p_\kappa(z_{01}), \cdots, p_1(z_{1d}), \cdots, p_\kappa(z_{1d})]$$

- Regress D_i linearly on $P_\kappa(z_0, z_1)$ to get $\hat{P}_i(z_0, z_1)$
- Lemma: Under regularity assumptions in Belloni et al. (2015),

$$\max_{g=1,\ldots,G} \left| \hat{P}_i \left(Z_{0g}, Z_{1g} \right) - P_i \left(Z_{0g}, Z_{1g} \right) \right| = O_p \left[\sqrt{\frac{\kappa \log \kappa}{G}} + \kappa^{-s} \right],$$

$\kappa \to \infty$ as $G \to \infty$, $\kappa^{m/(m-2)} \log \kappa / G = O(1)$ for any $m > 2$,
s: exponent of the Hölder condition

Second stage: Estimate marginal treatment response

Step 1: Estimate the denominator of estimand

$$\frac{\partial^2 \mathbb{E}\left[D_0 D_1 \mid P_0 = p_0, P_1 = p_1\right]}{\partial p_0 \partial p_1}$$

- Conducting a local polynomial regression of order three with bandwidth h_{G1}

$$\min_{b_0, \cdots, b_9} \sum_{g=1}^{G} \left[D_{0g} D_{1g} - b_0 - b_1(\hat{P}_{0g} - p_0) - \cdots b_4(\hat{P}_{0g} - p_0)(\hat{P}_{1g} - p_1)\right.$$
$$\left. - \cdots - b_9(\hat{P}_{1g} - p_1)^3\right]^2 K_{h_{G1}}\left(\hat{P}_g - p\right)$$

$\hat{b}_4$ estimates $\partial^2 \mathbb{E}[D_0 D_1 \mid P_0 = p_0, P_1 = p_1] / \partial p_0 \partial p_1$

Second stage: Estimate marginal treatment response

Step 2: Estimate the numerator of estimand

$$\frac{\partial^2 \mathbb{E}\left[Y_{idd'} \mid P_0 = p_0, P_1 = p_1\right]}{\partial p_0 \partial p_1}$$

- Conducting a local polynomial regression with bandwidth h_{G2}

$$\min_{c_0,\cdots,c_9} \sum_{g=1}^{G} \left[Y_{idd'} - c_0 - c_1(\hat{P}_{0g} - p_0) - \cdots c_4(\hat{P}_{0g} - p_0)(\hat{P}_{1g} - p_1) - \cdots - c_9(\hat{P}_{1g} - p_1)^3\right]^2 K_{h_{G2}}\left(\hat{P}_g - p\right)$$

$\hat{c}_4$ estimates $\partial^2 \mathbb{E}[Y_{idd'} \mid P_0 = p_0, P_1 = p_1]/\partial p_0 \partial p_1$

Step 3: Estimate marginal treatment response $m_i^{(d,d')}(p_0, p_1)$

$$\hat{m}_i^{(d,d')}(p_0, p_1) = \frac{\hat{c}_4}{\hat{b}_4}$$

Back to Estimation

Exposure to function of peer treatments

$$\begin{cases} Y_{ig} = Y_{ig}\left(1, H_g\right) D_{ig} + Y_{ig}\left(0, H_g\right)\left(1 - D_{ig}\right) \\ D_{ig} = \mathbb{1}\left\{V_{ig} \leq h(Z_{ig})\right\} \\ H_g = m\left(Z_g, \varepsilon_g\right) \end{cases}$$

- Group size n_g can be large and heterogeneous
  - E.g., groups can be defined as villages
- $Z_g \in \mathbb{R}^k$ is instrument randomly assigned to groups: the proportion of treated children in a cash transfer program
- $Z_{ig} \in \mathbb{R}^l$ is instrument received by unit i: whether the child i receives cash transfer and the proportion of cash transfer assignment
- Treatment $D_{ig} \in \{0, 1\}$ depends on Z_{ig} and individual unobservable V_{ig}: School dropout depends on cash transfer assignment and individual ability

Exposure to function of peer treatments

$$\begin{cases} Y_{ig} = Y_{ig}\left(1, H_g\right) D_{ig} + Y_{ig}\left(0, H_g\right)\left(1 - D_{ig}\right) \\ D_{ig} = \mathbb{1}\left\{V_{ig} \leq h(Z_{ig})\right\} \\ H_g = m\left(Z_g, \varepsilon_g\right) \end{cases}$$

- $H_g : \boldsymbol{D}_g \mapsto \mathbb{R}$ is a known continuous exposure mapping
  - E.g., $H_g = \sum_{i=1}^{n_g} D_{ig}/n_g$ is the dropout proportion in a village
  - Express H_g as a reduced-form function of (Z_g, ε_g) Example
- $\varepsilon_g \in \mathbb{R}$ is a continuous group-level unobservable
  - E.g., ε_g captures the group's unobserved homophilic preference
  - No restrictions on correlation between V_{ig} and ε_g
- Outcome $Y_{ig} \in \mathbb{R}$ depends on D_{ig} and H_g
  - E.g., individual's long-term outcome depends on her dropout status and the village dropout rate

Exposure setting assumptions

Additional assumption

4. (Monotonicity) $m(z, e)$ is continuous and strictly monotonic in e given z

- E.g., the village dropout rate monotonically decreases with the group preference for attending high school, given instrument values
- $\varepsilon_g = m_z^{-1}(H_g)$ by inverting $H_g = m(z, \varepsilon_g)$ w.r.t. ε_g given $Z_g = z$
- Propensity score function identifies $m_{Z_g}^{-1}(H_g)$
- Identify MCSEs and MCDEs using propensity scores as control functions

Back to main theorem

Explicit reduced function of exposure level

Suppose that $H_g = \sum_{i=1}^{n_g} D_{ig}/n_g$ and $D_{ig} = \mathbb{1}\{V_{ig} \leq h(Z_g)\}$

Two types of individuals, $\mathcal{I}_g$: the set of individual indices in group g

- Type 1: $V_{ig} = \varepsilon_g, i \in \mathcal{I}_g^1 \subseteq \mathcal{I}_g$
- Type 2: $V_{jg} = 1 - \varepsilon_g, j \in \mathcal{I}_g^2 = \mathcal{I}_g \setminus \mathcal{I}_g^1$
- $|\mathcal{I}_g^1|/|\mathcal{I}_g| = \varepsilon_g$
- $\varepsilon_g \in (0, 1)$: (i) captures the unobserved heterogeneity among group members; (ii) reflects the proportion of individual type

$$
\begin{aligned}
H_g =& \frac{1}{n_g} \sum_{i=1}^{n_g} D_{ig} = \frac{1}{n_g} \sum_{i \in \mathcal{I}_g^1} D_{ig} + \frac{1}{n_g} \sum_{i \in \mathcal{I}_g^2} D_{ig} \\
=& \varepsilon_g \mathbb{1}\{\varepsilon_g \leq h(Z_g)\} + (1 - \varepsilon_g)\mathbb{1}\{1 - \varepsilon_g \leq h(Z_g)\} \equiv m(Z_g, \varepsilon_g)
\end{aligned}
$$

Generally, $m(\cdot, \cdot)$ is an unknown reduced-form function, ε_g summarizes the group's unobserved characteristics Back to Extension: Model

Identification in exposure setting

- Identify threshold function $h_i(\cdot)$

$$P_{ig}(z) \equiv \mathbb{P}\left(D_{ig} = 1 \mid Z_g = z\right)$$
$$= \mathbb{P}\left(V_{ig} \leq h_i\left(Z_g\right) \mid Z_g = z\right) = h_i(z)$$

- Identify the inverse of $m(\cdot)$

$$P_g\left(Z_g, h\right) \equiv \mathbb{P}\left(H_g \leq h \mid Z_g = z\right)$$
$$= \mathbb{P}\left(\varepsilon_g \leq m_z^{-1}(h) \mid Z_g = z\right) = m_z^{-1}(h)$$

Identification in exposure setting

- Identify conditional distribution of $V_{ig} \mid \varepsilon_g$

$$
\begin{aligned}
&\mathbb{P}\left(D_{ig} = 1 \mid H_g = h, P_{ig}\left(Z_g\right) = p_0, P_g\left(Z_g, H_g\right) = p_1\right) \\
=&\mathbb{P}\left(V_{ig} \leq p_0 \mid \varepsilon_g = m_{Z_g}^{-1}(h), h_i\left(Z_g\right) = p_0, m_{Z_g}^{-1}(h) = p_1\right) \\
=&\mathbb{P}\left(V_{ig} \leq p_0 \mid \varepsilon_g = p_1\right)
\end{aligned}
$$

- Identify the density $f_{V_{ig}|\varepsilon_g = p_1}\left(p_0\right)$ as

$$
\frac{\partial}{\partial p_0}\mathbb{P}\left(D_{ig} = 1 \mid H_g = h, P_{ig}\left(Z_g\right) = p_0, P_g\left(Z_g, H_g\right) = p_1\right)
$$

if $\mathbb{P}\left(D_{ig} = 1 \mid \cdot, \cdot, \cdot\right)$ is differentiable

Identification in exposure setting

- Define the marginal treatment response function as $\mathbb{E}\left[Y_{ig}(d,h) \mid V_{ig} = p_0, \varepsilon = p_1\right]$
  - Take $\mathbb{E}\left[Y_{ig}(1,h) \mid V_{ig} = p_0, \varepsilon_g = p_1\right]$ as example

$$
\begin{aligned}
&\mathbb{E}\left[Y_{ig}D_{ig} \mid H_g = h, P_{ig}\left(Z_g\right) = p_0, P_g\left(Z_g, H_g\right) = p_1\right] \\
=&\mathbb{E}\left[Y_{ig}\mathbb{1}\left\{V_{ig} \leq p_0\right\} \mid \varepsilon_g = p_1\right]
\end{aligned}
$$

  - If $\mathbb{E}\left[Y_{ig}D_{ig} \mid \cdot, \cdot, \cdot\right]$ is differentiable

$$
\begin{aligned}
&\frac{\partial}{\partial p_0}\mathbb{E}\left[Y_{ig}D_{ig} \mid H_g = h, P_{ig}\left(Z_g\right) = p_0, P_g\left(Z_g, H_g\right) = p_1\right] \\
=&\mathbb{E}\left[Y_{ig}(1,h) \mid V_{ig} = p_0, \varepsilon_g = p_1\right] \cdot f_{V_{ig}\mid\varepsilon_g = p_1}\left(p_0\right)
\end{aligned}
$$

- MCSEs and MCDEs are identified from marginal treatment response functions

Back to Extension: Identification

Parametric Procedure: Estimation Bias

Estimation Bias				
	(0.4,0.6)	(0.5,0.5)	(0.6,0.4)	ρ
Panel A1: MCDE (n=1000)				
D = 1	0.106	0.100	0.075	-0.001
D = 0	0.007	-0.030	-0.072	
Panel A2: MCSE (n=1000)				
D = 1	0.036	0.078	0.121	-0.001
D = 0	-0.064	-0.051	-0.026	
Panel B1: MCDE (n=10000)				
D = 1	0.083	0.056	0.034	-0.0007
D = 0	0.017	-0.010	-0.041	
Panel B2: MCSE (n=10000)				
D = 1	0.030	0.048	0.076	-0.0007
D = 0	-0.036	-0.018	0.001	

Note: Monte Carlo simulations are repeated 500 times.

Parametric Procedure: Coverage Rates

	95% Confidence Interval Coverage Rate			
	(0.4,0.6)	(0.5,0.5)	(0.6,0.4)	ρ
	Panel A1: MCDE (n=1000)			
D = 1	0.952	0.964	0.972	0.948
D = 0	0.958	0.962	0.96	
	Panel A2: MCSE (n=1000)			
D = 1	0.962	0.972	0.966	0.948
D = 0	0.968	0.956	0.954	
	Panel B1: MCDE (n=10000)			
D = 1	0.938	0.948	0.954	0.94
D = 0	0.936	0.948	0.946	
	Panel B2: MCSE (n=10000)			
D = 1	0.95	0.946	0.94	0.94
D = 0	0.952	0.95	0.954	

Note: CIs are based on 500 bootstrap replications. Monte Carlo simulations are repeated 500 times.

Back to parametric procedure